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Monotonicity properties of interval solutions and the Dutta-Ray solution for convex interval games

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Abstract

This paper examines several monotonicity properties of value-type interval solutions on the class of convex interval games and focuses on the Dutta-Ray (DR) solution for such games. Well known properties for the classical DR solution are extended to the interval setting. In particular, it is proved that the interval DR solution of a convex interval game belongs to the interval core of that game and Lorenz dominates each other interval core element. Consistency properties of the interval DR solution in the sense of Davis-Maschler and of Hart-Mas-Colell are verified. An axiomatic characterization of the interval DR solution on the class of convex interval games with the help of bilateral Hart-Mas-Colell consistency and the constrained egalitarianism for two-person interval games is given.

JEL Classification: C71

Keywords: cooperative interval games, convex games, the constrained egalitarian solution, the equal division core, consistency

1 Introduction

Cooperative interval games are introduced and studied in Alparslan Gök, Miquel and Tijs (2008) and Alparslan Gök, Branzei and Tijs (2008a,b), where the interval core plays a central role. Such games model situations with cooperation with incomplete information of agents and of their coalitions about the payoffs they can obtain for sure.

However, there are many real-life situations in which people or businesses are uncertain about their coalitional payoffs. Situations with uncertain payoffs in which the agents cannot await the realizations of their coalition payoffs cannot be modeled according to classical game theory. Several models that are useful to handle uncertain payoffs exist in the game theory literature. We refer here to chance-constrained games (Charnes and Granot (1973)), cooperative games with stochastic payoffs (Suijs et al. (1999)), cooperative games with random payoffs (Timmer, Borm and Tijs (2005)). In all these models stochastics plays an important role.

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This paper deals with a model of cooperative games where only bounds for payoffs of coalitions are known with certainty. Such games are called cooperative interval games. Formally, a *cooperative interval game* in coalitional form (Alparslan Gök, Miquel and Tijs (2008)) is an ordered pair $\langle N, w\rangle$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all nonempty, compact intervals in $\mathbb{R}$. For each $S \in 2^N$, the worth set (or worth interval) $w(S)$ of the coalition S in the interval game $\langle N, w\rangle$ is of the form $[\underline{w}(S), \overline{w}(S)]$. We denote by IG^N the family of all interval games with player set N. Note that if all the worth intervals are degenerate intervals, i.e. $\underline{w}(S) = \overline{w}(S)$ for each $S \in 2^N$, then the interval game $\langle N, w\rangle$ corresponds in a natural way to the classical cooperative game $\langle N, v\rangle$ where $v(S) = \underline{w}(S)$ for all $S \in 2^N$. Some classical TU-games associated with an interval game $w \in IG_N$ will play a key role, namely the *border* games $\langle N, \underline{w}\rangle$, $\langle N, \overline{w}\rangle$ and the *length* game $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$. An interval solution concept $\mathcal{F}$ on IG_N is a map assigning to each interval game $\langle N, w\rangle \in IG_N$ a set of n-dimensional vectors whose components belong to $I(\mathbb{R})$. We denote by $I(\mathbb{R})^N$ the set of all such interval payoff vectors. Cooperative interval games are very suitable to describe real-life situations in which people or firms that consider cooperation have to sign a contract when they cannot pin down the attainable coalition payoffs, knowing with certainty only their lower and upper bounds. The contract should specify how the players' payoff shares will be obtained when the uncertainty of the worth of the grand coalition is removed at an ex post stage. In the following we briefly explain how interval solutions for cooperative interval games are useful to support decision making regarding cooperation and related binding contracts. A vector interval allocation obtained by an agreed upon solution concept offers at the ex ante stage an estimation of what individual players may receive, between two bounds, when the uncertainty on the reward of the grand coalition is removed in the ex post stage. We notice that the agreement on a particular interval allocation $(I_1, I_2, \ldots, I_n)$ based on an interval solution concept merely says that the payoff x_i that player i will receive in the interim or ex post stage is in the interval I_i. This is a very weak contract to settle cooperation. Therefore, writing down in the contact the protocol to be used when the uncertainty on $w(N)$ is removed at the ex post stage, is compulsory. Such protocols are described in Branzei, Tijs and Alparslan Gök (2008).

The first step in the study of interval game solutions is to extend classical theory of cooperative game solutions to interval games. For example, we can apply some single-valued solution concept to both border games, and in the case when the solution of the upper game weakly dominates that of the lower game, the corresponding interval vector could be admitted as the interval solution, *generated* by a classical cooperative game solution. Just in this manner the interval Shapley value for convex interval games was defined in Alparslan Gök, Branzei and Tijs (2008b). The same approach can be applied to the extension of set-valued solutions as well Alparslan Gök, Branzei and Tijs (2008a,b).

Naturally, the problem of existence of such interval solution arises. In fact if for some interval game $\langle N, w\rangle$ the characteristic function values of the lower and upper games on the grand coalition coincide: $\underline{w}(N) = \overline{w}(N)$, then for any single-valued classical solution φ the (vector) inequality $\varphi(N, \underline{w}) \leq \varphi(N, \overline{w})$ is impossible, and this approach cannot be applied to the extension of the solution φ to the interval game $\langle N, w\rangle$.

It is clear that the possibility of the extension of a classical cooperative game solution to interval games depends both on the class of interval games into consideration and on monotonicity properties of the classical cooperative game solution itself. Thus, in the papers Alparslan Gök, Branzei and Tijs (2008a,b) the class of convex interval games was introduced. It turned out that the most known cooperative game solutions such that the core, the Shapley value, and the Weber set are extendable to the class of interval convex games (though, as for the classical case, they exist on larger classes of interval games).

This paper examines different monotonicity properties of classical cooperative game solutions on the class of convex games and with the help of these properties verifies the existence or not existence of the corresponding interval game solutions.

A special attention is devoted to the extension of the (constrained egalitarian) Dutta–Ray solution (Dutta (1990), Dutta and Ray (1989)) to the interval setting. It is shown that this solution exists on the class of convex interval games, belongs to the interval core, and has the same monotonicity properties as the classical Dutta–Ray solution. The last one has two nice axiomatic characterizations on the class of convex TU games both with the consistency axioms: one uses the Davis–Maschler consistency, and another uses the Hart–Mas-Colell consistency. It turns out that the interval Dutta–Ray solution has only one characterization with the help Hart–Mas-Colell consistency, because the Davis–Maschler reduced interval game may not belong to the class of convex interval games.

The outline of the paper is as follows. In Section 2 we recall basic definitions, notations and results on (convex) interval games. In Section 3 we recall the known monotonicity properties of TU game solutions and connect them with the existence of the corresponding generated interval solutions and the inheritance by them of the monotonicity properties. In Section 4 we prove that the interval Dutta-Ray solution of a convex interval game belongs to the interval core of the game. The Lorenz domination on the product vector set is determined and it is shown that the interval Dutta–Ray solution Lorenz dominates each other interval core element. Section 5 provides an axiomatic characterization of the Dutta-Ray solution for convex interval games. We conclude in Section 6 with remarks about alternative ways to define and axiomatically characterize the (Dutta-Ray) constrained egalitarian solution on the class of convex interval games.

2 Definitions and Notation

An *interval game* is a triple $\langle N, (\underline{w}, \overline{w})\rangle$ where N is a finite set of players, $\underline{w}, \overline{w} : 2^N \to \mathbb{R}$ are a *lower* and a *upper* characteristic functions, respectively, such that for each coalition $S \subset N$, $\underline{w}(S) \leq \overline{w}(S)$. The TU games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ are called the *lower* and the *upper* games of the interval game $\langle N, (\underline{w}, \overline{w})\rangle$, respectively.

Let G_N be an arbitrary class of TU games with the player set N. Further we denote by IG_N the class of interval games with the player set N such that for any $\langle N, (\underline{w}, \overline{w})\rangle \in IG_N$ both the lower and upper games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ belong to the class G_N.

Denote by $X(N, \underline{w}), X(N, \overline{w})$ the sets of feasible payoff vectors of the lower and upper games, and by $Y(N, \underline{w}), Y(N, \overline{w})$ the sets of *efficient* payoff vectors, respectively:

$$X(N, \underline{w}) = \{x \in \mathbb{R}^N \mid \textstyle\sum_{i\in N} x_i \leq \underline{w}(N)\},$$
$$X(N, \overline{w}) = \{x \in \mathbb{R}^N \mid \textstyle\sum_{i\in N} x_i \leq \overline{w}(N)\},$$

$$Y(N, \underline{w}) = \{x \in X(N, \underline{w}) \mid \textstyle\sum_{i\in N} x_i = \underline{w}(N)\},$$
$$Y(N, \overline{w}) = \{x \in X(N, \overline{w}) \mid \textstyle\sum_{i\in N} x_i = \overline{w}(N)\},$$

Definition 1 A *single-valued solution (value)* ϕ for a class IG_N of interval games is a mapping assigning to each interval game $\langle N, (\underline{w}, \overline{w})\rangle \in IG'_N$ a pair of vectors $\phi(N, (\underline{w}, \overline{w})) = (x, y) \in \mathbb{R}^{2n}$ such that $x \in X(N, \underline{w}), y \in X(N, \overline{w})$ and $x \leq y$.

Definition 2 An interval value ϕ on a class of interval games IG_N is *generated by a TU game value* φ if

$$\phi(N, (\underline{w}, \overline{w})) = (\varphi(N, \underline{w}), \varphi(N, \overline{w})). \tag{1}$$

Equality (1) implies that the inequality

$$\varphi(N, \underline{w}) \leq \varphi(N, \overline{w}) \tag{2}$$

should hold, and, hence, not all TU game values can be extended to the generated interval values, and even if a value can be extended, then only for some special classes of TU and interval games.

In the sequel we consider only interval values generated by some known TU game values.

Consider the class G_N^c of convex TU games with a finite set of players N, Define the class $IG_{\mathcal{N}}^c$ of *convex interval games* with the universal set of players $\mathcal{N}$ by the following way:

$$\langle N, (\underline{w}, \overline{w})\rangle \in \mathcal{IG}_{\mathcal{N}}^c \iff \langle N, \overline{w}\rangle, \langle N, \underline{w}\rangle, \langle N, \overline{w} - \underline{w}\rangle \in G_N^c \text{ and } \underline{w}(S) \leq \overline{w}(S) \text{ for all } S \subset N.$$

Given a vector $x \in \mathbb{R}^N$ and a coalition $S \subset N$, by x_S we denote the projection of the vector x on the subspace $\mathbb{R}^S$, and by $x(S)$ the sum $x(S) = \sum_{i \in S} x_i$.

The set of intervals of the real line we denote by $I(\mathbb{R})$, and the set of $|R|$-dimensional interval vectors we denote by $I(\mathbb{R})^N$. An interval $[a_1, a_2]$ *dominates* an interval $[b_1, b_2]$, $[a_1, a_2] \succcurlyeq [b_1, b_2]$ if $a_1 \geq b_1, a_2 \geq b_2$. An interval vector $\mathbf{a} = ([a_1, a_1'], \ldots, [a_n, a_n'])$ *dominates* an interval vector $\mathbf{b} = ([b_1, b_1'], \ldots, [b_n, b]_n])$, $\mathbf{a} \succcurlyeq \mathbf{b}$, if $[a_i, a_i'] \succcurlyeq [b_i, b_i']$ for $i = 1, \ldots, n$.

In the next section we show which TU game values for convex games can be extended to the generated interval values and which ones can not.

By $C(N, v)$ we denote the core of $\langle N, v\rangle$, and by $\mathcal{C}(N, w)$ the *interval core* (of the interval game $\langle N, (\underline{w}, \overline{w})\rangle, w = (\underline{w}, \overline{w})$):

$$\mathcal{C}(N, w) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in C(N, \underline{w}), y \in C(N, \overline{w}), x \leq y\}.$$

We notice that this definition is different than the usual one, which regards the interval core as a set of $|N|$- dimensional vectors in $I(\mathbb{R})^N$, but it is equivalent in its consequences.

3 Monotonicity properties of TU game values and of the corresponding generated interval values

3.1 Existence of interval values generated by TU game values

In this section we consider the interval values on the class of convex interval games IG_N^c.

Given a TU value φ for the class G_N^c, the existence of the generated by it interval value ϕ on IG_N^c, i.e. the fulfilment of inequality (2) is equivalent to the following monotonicity property of φ :

Convex monotonicity (CvM). If $\langle N, v\rangle$, $\langle N, v'\rangle$, $\langle N, v' - v\rangle \in G_N^c$, and $v'(S) \geq v(S)$ for all $S \subset N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Let us compare this property with other known monotonicity properties of TU game solutions[1]:

[1] The definitions of the properties are given for arbitrary classes of TU games, so they are not indicated.

Aggregate monotonicity If $v'(N) > v(N)$ and $v'(S) = v(S)$ for all $S \subsetneq N$, then $\varphi(N, v') \geq \varphi(N, v)$.

Coalitional monotonicity. For each coalition $S \subset N$, $v'(S) > v(S)$ and $v'(T) = v(T)$ for all $T \neq S$ imply $\varphi_i(N, v') \geq \varphi_i(N, v)$ for all $i \in S$.

Contribution monotonicity (CM). For each $i \in N$ inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ for all $S \not\ni i$ imply $\varphi_i(N, v') \geq \varphi_i(N, v)$.

Weak contribution monotonicity (WCM) (Hokari, van Gellekom 2002) If for all $i \in N$ and all coalitions $S \not\ni i$ the inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ hold, then $\varphi(N, v') \geq \varphi(N, v)$.

Note that all these properties were defined for games with the same sets of players. It is clear that

$$CM \implies WCM \implies AM. \tag{3}$$

Let us check where convex monotonicity is placed in relations (3).

Proposition 1 *On the class of convex games* $G^c_{\mathcal{N}}$

$$WCM \implies CvM \implies AM.$$

Proof. Let $\langle N, v\rangle, \langle N, v'\rangle, \langle N, v'-v\rangle$ be convex games such that $v'(S) \geq v(S)$ for all $S \subset N$. Then for all $i \in N$ and $S \not\ni i$

$$v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S). \tag{4}$$

If a value φ on $G^c_{\mathcal{N}}$ satisfies weak contribution monotonicity, then $\varphi(N, v') \geq \varphi(N, v)$, and φ satisfies convex monotonicity.

Let now φ' be any value on the class $\mathcal{G}^c_{\mathcal{N}}$ that satisfies convex monotonicity. Then for games $\langle N, v\rangle, \langle N, v'\rangle$ inequalities (4) hold, inclusively for those such that $v(S) = v'(S)$ for all $S \subsetneq N$, $v'(N) > v(N)$, implying $\varphi(N, v') \not\geq \varphi(N, v)$. $\square$

Relations (3) and Proposition 1 permit to check for what TU game values for convex games the generated interval values exist or not.

It is well-known that the Shapley value satisfies contribution monotonicity. Therefore, there exists the interval Shapley value on the class of convex interval games (Alparslan Gök, Branzei and Tijs (2008b)).

On the other hand, it is known that the prenucleolus and the τ-value on the class of convex games do not satisfy aggregate monotonicity (Hokari (2000), Hokari and van Gellekom (2002)). Therefore, the interval prenucleolus and the interval τ-value do not exist on the class $IG^c_{\mathcal{N}}$.

The (constrained) egalitarian solution for TU games was defined by Dutta and Ray (1989) as the unique Lorenz maximal allocation in the Lorenz core. We call it the it Dutta–Ray solution (DR). This solution can be empty, its existence was proved in the same paper for the class of convex games. For each convex game $\langle N, v\rangle$ the Dutta–Ray solution is the unique allocation in the core which Lorenz dominates all other core allocations. This solution was characterized on the class of convex games by Dutta (1990) in two ways, both using consistency: he proved that the DR is the unique solution satisfying constrained egalitarianism (CE) for two-person games and consistency either in the definition due to Davis and Maschler (1965), or in the definition due to Hart and Mas-Colell (1989).

The Dutta–Ray solution on the class of convex TU games possesses many attractive properties. In particular, Hokari and van Gellekom (2002) proved that the DR solution over the class of convex games satisfies *weak contribution monotonicity*, hence, by Proposition 1 it satisfies convex monotonicity providing the existence of the generated Dutta–Ray interval solution on the class of convex interval games.

The properties and a characterization of the interval Dutta–Ray solution will be the main subject of the next sections.

On the other hand, it is known that the prenucleolus and the τ-value on the class of convex games do not satisfy aggregate monotonicity (Hokari (2000), Hokari and van Gellekom (2002)). Therefore, the interval prenucleolus and the interval τ-value do not exist on the class IG_N^c.

The last monotonicity property compares players' payoffs with respect to solution vectors in the initial game and its subgames:

Population monotonicity. If $\langle N, v\rangle$ is a convex game and $N' \subset N$, then $\varphi_i(N,v) \geq \varphi_i(N',v)$ for all $i \in N'$, where $\langle N', v\rangle$ is the subgame of $\langle N, v\rangle$.

This property assures the existence of population monotonic allocation schemes, (Sprumont (1990)). Recall that for a game $v \in G^N$ a scheme $a = (a_{iS})_{i\in S, S\in 2^N\setminus\{\emptyset\}}$ of real numbers is a population monotonic allocation scheme of v if

(i) $\sum_{i\in S} a_{iS} = v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $a_{iS} \leq a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

We notice that convexity of v is a sufficient condition for the the existence of population monotonic allocation schemes.

3.2 Inheritance of monotonicity properties by interval values

It is not difficult to extend the above defined monotonicity properties (except for convex monotonicity) to interval values. For interval values we demand that the properties hold both for lower and upper games. Let ϕ be an interval value for the class $IG_{\mathcal{N}}^c$ of interval convex games. The following definitions are the extensions to interval convex games of the given above monotonicity properties of TU game values.

Aggregate monotonicity If $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ are interval convex games such that $\underline{w}(S) = \underline{w'}(S), \overline{w}(S) = \overline{w'}(S)$ for all $S \subsetneqq N$, and $\underline{w'}(N) > \underline{w}(N), \overline{w'}(N) > \overline{w}(N)$, then $\phi(N, v') \succcurlyeq \phi(N, v)$.

Coalitional monotonicity. If for interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ for some coalition $S \subset N$ the following inequalities hold: $\underline{w'}(S) > \underline{w}(S), \overline{w'}(S) > \overline{w}(S)$ and $\underline{w'}(T) = \underline{w}(T), \overline{w'}(T) = \overline{w}(T)$ for all $T \neq S$, then $\phi_i(N, (\underline{w'}, \overline{w'})) \succcurlyeq \phi_i(N, (\underline{w}, \overline{w}))$ for all $i \in S$.

Contribution monotonicity (CM). For interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$ and for each $i \in N$ inequalities
$\underline{w'}(S\cup\{i\}) - \underline{w'}(S) \geq \underline{w}(S\cup\{i\}) - \underline{w}(S)$, $\overline{w'}(S\cup\{i\}) - \overline{w}(S) \geq \overline{w}(S\cup\{i\}) - \overline{w}(S)$ for all $S \not\ni i$ imply $\phi_i(N, v') \succcurlyeq \phi_i(N, v)$.

Weak contribution monotonicity If for interval convex games $\langle N, (\underline{w}, \overline{w})\rangle$ and $\langle N, (\underline{w'}, \overline{w'})\rangle$, for all $i \in N$, and all coalitions $S \not\ni i$ the inequalities $\underline{w'}(S\cup\{i\}) - \underline{w'}(S) \geq \underline{w}(S\cup\{i\}) - \underline{w}(S)$, $\overline{w'}(S\cup\{i\}) - \overline{w'}(S) \geq \overline{w}(S\cup\{i\}) - \overline{w}(S)$ hold, then $\phi(N, v') \succcurlyeq \phi(N, v)$.

Population monotonicity. If $\langle N, (\underline{w}, \overline{w})\rangle$ is an interval convex game and $N' \subset N$, then $\phi_i(N, (\underline{w}, \overline{w})) \succcurlyeq \phi_i(N', (\underline{w}, \overline{w}))$ for all $i \in N'$, where $\langle N', (\underline{w}, \overline{w})\rangle$ is the subgame of $\langle N, (\underline{w}, \overline{w})\rangle$.

From the definitions it follows that all these properties are inherited by interval values generated by TU game values: if a value φ on the class of TU convex games G_N^c satisfies one of the monotonicity properties, then the generated interval value ϕ on the class IG_N^c satisfies the same interval property.

In particular, since the Shapley value and the Dutta–Ray solution on the class of convex games are population monotonic, we obtain that the interval Shapley value and the interval Dutta–Ray solution are population monotonic on the class of interval convex games as well.

This last monotonicity property provides the existence of interval population monotonic allocation schemes (Alparslan Gök, Branzei and Tijs (2008b)). Recall that for a game $w \in IG^N$ a scheme $A = (A_{iS})_{i\in S, S\in 2^N\setminus\{\emptyset\}}$ with $A_{iS} \in I(\mathbb{R})^N$ is an interval monotonic allocation scheme of w if

(i) $\sum_{i\in S} A_{iS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $A_{iS} \preccurlyeq A_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

We notice that convexity of w is a sufficient condition for the existence of interval population monotonic allocation schemes.

4 The interval Dutta–Ray solution on the class of convex interval games

4.1 Properties of the interval Dutta–Ray solution

The solution CE of *constrained egalitarianism* on the class of two-person superadditive games is defined for each game $\langle\{i, j\}, v\rangle$ as follows:

$$CE_i(\{i,j\}, v) = \begin{cases} \frac{v(\{i,j\})}{2} & \text{if } \frac{v(i,j)}{2} \geq \max\{v(\{i\}), v(\{j\})\}, \\ v(\{i\}), & \text{if } \frac{v(\{i,j\})}{2} < v(\{i\}), v(\{i\}) > v(\{j\} \\ v(\{1,2\}) - v(\{2\}), & \text{if } \frac{v(\{1,2\})}{2} \leq v(\{2\}), v(\{j\}) > v(\{i\}). \end{cases} \tag{5}$$

Definition (5) shows that the CE solution assigns to each two-person superadditive game the payoff vector in the core nearest to the diagonal, i.e. to the equal share efficient payoff vector. This solution vector Lorenz dominates all other vectors from the core: $CE(\{i,j\}, v) \succ_{Lor} x$ for all $x \in C(\{i,j\}, v) \setminus CE(\{i,j\}, v)$. Recall that if we consider a society of n individuals with aggregate income fixed at I units, and for any $x \in \mathbb{R}^n_+$ denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained by rearranging its coordinates in a non-decreasing order, that is, $\hat{x}_1 \leq \hat{x}_2 \leq \ldots \leq \hat{x}_n$ then the Lorenz domination relation is defined as follows. For any $x, y \in \mathbb{R}^n_+$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = I$, we say that x *Lorenz dominates* y, and denote it by $x \succ_{Lor} y$, if and only if $\sum_{i=1}^p \hat{x}_i \geq \sum_{i=1}^p \hat{y}_i$ for all $p \in \{1, \ldots, n-1\}$, with at least one strict inequality.

The *Dutta-Ray solution* extends the CE solution to all convex TU games: it assigns to each convex game $\langle N, v\rangle \in \mathcal{G}_N^c$ the vector $DR(N, v) \in C(N, v)$ which Lorenz dominates all other vectors from the core:

$$DR(N, v) \succ_{Lor} x \text{ for all } x \in C(N, v).$$

Proposition 1 permits to define the interval Dutta–Ray solution for interval convex games as a mapping assigning to each convex interval game $\langle N, (\underline{w}, \overline{w})\rangle$ the pair of vectors $(DR(N, \underline{w}, DR(N, \overline{w}))$. This definition can be done in the form of the Lorenz domination property as that for convex TU games. For this, first we should extend the Lorenz domination to sets of ordered pairs of vectors $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $x \leq y$.

Let $A = \{(x, y) \,|\, x \in \mathbb{R}^N, y \in \mathbb{R}^N, x \leq y\}$ be a set of pairs of vectors, $(x, y), (x', y') \in A$. We say that (x, y) *Lorenz dominates* (x', y'), if the *Lorenz curve* $L(x, y)$ Pareto dominates the Lorenz curve $L(x', y')$. Note that in a weakly increasing ordering of the vector (x, y) defining the Lorenz curve $L(x, y)$, it may happen that $x_i > y_j$ for some components $i > j$.

Proposition 2 *For any interval convex game $\langle N, w\rangle = \langle N, (\underline{w}, \overline{w})\rangle \in IG^c_N$ the interval Dutta–Ray solution $(DR(N, \underline{w}), DR(N, \overline{w}))$ belongs to the interval core $\mathcal{C}(N, w)$ and Lorenz dominates all other vectors $(x, y) \in \mathcal{C}(N, w)$.*

Proof. Since $DR(N, \underline{w}) \in C(N, \underline{w}), DR(N, \overline{w}) \in C(N, \overline{w})$ and $DR(N, \underline{w}) \leq DR(N, \overline{w})$ we have $(DR(N, \underline{w}), DR(N, \overline{w})) \in \mathcal{C}(N, w)$.

By the definition of the DR solution on the class G^c_N

$$DR(N, \underline{w}) = x^* \succ_{Lor} x, \;\; DR(N, \overline{w}) = y^* \succ_{Lor} y \text{ for all } x \in C(N, \underline{w})\backslash\{x^*\}\,, y \in C(N, \overline{w})\backslash\{y^*\}\,.$$

Then, by separability of the Lorenz domination,

$$\begin{array}{l} (x^*, y) \succ_{Lor} (x, y) \text{ for any } y \in \mathbb{R}^N \text{ and all } x \in C(N, \underline{w}) \setminus \{x^*\}\,, \\ (x, y^*) \succ_{Lor} (x, y) \text{ for any } x \in \mathbb{R}^N \text{ and all } y \in C(N, \overline{w}) \setminus \{y^*\}\,. \end{array} \tag{6}$$

It is clear that relations (6) imply the demanded result. □

The DR solution on the class G^c_N is covariant with respect to identical affine transformations of players utilities. It means that for any game $\langle N, v\rangle \in G^c_N$, any positive number $\alpha \in \mathbb{R}_+$, and arbitrary vector $\bar{b} = (b, b \ldots, b) \in \mathbb{R}^N$, it holds

$$DR(N, \alpha v + \bar{b}) = \alpha DR(N, v) + \bar{b}, \tag{7}$$

where for all $S \subset N$, $(\alpha v + \bar{b})(S) = \alpha v(S) + b|S|$.

It turns out that this property is extended to the interval DR solution even in a stronger manner:

Proposition 3 *For any finite N the interval DR solution on the class IG^c_N is covariant with respect to identical affine transformations of players utilities, which may be different for lower and upper games: for arbitrary $\langle N, (\underline{w}, \overline{w})\rangle \in IG^c_N$, numbers $\alpha, \alpha' \in \mathbb{R}_+, \alpha \leq \alpha'$, and vectors $\bar{a} = (a, \ldots, a), \bar{b} = (b, \ldots, b) \in \mathbb{R}^N, a \leq b$, it holds*

$$DR(N, \alpha(\underline{w} + \bar{a}, \overline{w} + \bar{b})) = (\alpha DR(N, \underline{w} + \bar{a}), \alpha DR(N, \overline{w}) + \bar{b})),$$

and if the border games are positive, i.e. $\underline{w}(S) \geq 0$ for all $S \subset N$, then

$$DR(N, \alpha\underline{w} + \bar{a}, \alpha'\overline{w} + \bar{b}) = \alpha DR(N, \underline{w} + \bar{a}, \alpha' DR(N, \overline{w}) + \bar{b}).$$

Proof. First, notice that the pair $\langle N, \underline{w}+\bar{a}\rangle, \langle N, \overline{w}+\bar{b}\rangle$ defines the interval convex game $\langle N, (\underline{w} + \bar{a}, \overline{w} + \bar{b})$. In fact, both border games are convex, $\underline{w} + a \leq \overline{w} + b$ for all $S \subset N$, and the length game $\overline{w} - \underline{w} + \bar{b} - \bar{a}$ is also convex.

Similarly, it may be shown that $\langle N, (\alpha\underline{w} + \bar{a}, \alpha'\overline{w} + \bar{b})\rangle$ is a convex interval games, if $\underline{w}(S) \geq 0$ for all $S \subset N$.

Now equalities (7) establish the result. □

Similar to classical TU games, given an interval game $\langle N, (\underline{w}, \overline{w})\rangle$ we call the game $\langle N, \alpha(\underline{w} + \bar{a}, \overline{w} + \bar{b})\rangle$, where $\alpha > 0, \bar{a} \leq \bar{b}$, *strategically equivalent* to the game $\langle N, (\underline{w}, \overline{w})\rangle$. For the case of positive border games the game $\langle \alpha\underline{w} + \bar{a}, \alpha\overline{w} + \bar{b}\rangle$ is called *strategically equivalent* to the game $\langle N, (\underline{w}, \overline{w})\rangle$.

Recall Dutta's algorithm (Dutta (1990)) for the calculation of the DR solution for convex TU games: let for a convex TU game $\langle N, v\rangle$, $x = DR(N, v)$, and let the players be ordered with respect to their decreasing solution payoffs:

$$x = (\underbrace{a_1, \dots, a_1}_{T_1}, \underbrace{a_2, \dots, a_2}_{T_2}, \dots \underbrace{a_k \dots, a_k}_{T_k}). \tag{8}$$

The numbers $a_1 > a_2 > \dots, > a_k$ are found subsequently:

$$\begin{aligned} a_1 &= \max_{S \subset N} \tfrac{v(S)}{|S|} = \tfrac{v(T_1)}{|T_1|}, \\ \vdots &\;\; \vdots \\ a_j &= \max_{S \subset N \setminus \cup_{i=1}^{j-1} T_i} \tfrac{v^j(S)}{|S|} = \tfrac{v^j(T_j)}{|T_j|}, j = 2, \dots, k, \end{aligned} \tag{9}$$

where

$$v^j(S) = v(\bigcup_{i=1}^{j-1} T_i \cup S) - v(\bigcup_{i=1}^{j-1} T_i) \text{ for all } S \subset N \setminus \bigcup_{i=1}^{j-1} T_i.$$

It is clear that for finding the interval DR solution we should apply the algorithm for lower and upper games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle$ separately. Then, in the general case, the corresponding partitions of the player set N may be different for lower and upper games. However, it is clear that if the lower and the upper games are strategically equivalent then the partitions of N in coalitions whose players have equal shares corresponding to the DR solutions $DR(N, \underline{w}), DR(N, \overline{w})$ are the same. The analogous result holds for the interval DR solution:

Proposition 4 *Let two convex interval games $\langle N, (\underline{w}, \overline{w})\rangle, \langle N, (\underline{w}', \overline{w}')\rangle$ be strategically equivalent, and let $DR(N, (\underline{w}, \overline{w})) = (x, y)$ where*

$$x = DR(N, \underline{w}) = (\underbrace{a_1, \dots, a_1}_{T_1}, \underbrace{a_2, \dots, a_2}_{T_2}, \dots \underbrace{a_k \dots, a_k}_{T_k}),$$

$$y = DR(N, \overline{w}) = (\underbrace{b_1, \dots, b_1}_{Q_1}, \underbrace{b_2, \dots, b_2}_{Q_2} \dots \underbrace{b_r \dots, b_r}_{Q_r}),$$

and $a_1 > \dots > a_k$, $b_1 > \dots > b_r$. Then $DR(N, (\underline{w'}, \overline{w'})) = (x', y')$, where

$$x' = (\underbrace{a'_1, \dots, a'_1}_{T_1}, \underbrace{a'_2, \dots, a'_2}_{T_2}, \dots \underbrace{a'_k \dots, a'_k}_{T_k}),$$

$$y' = (\underbrace{b'_1, \dots, b'_1}_{Q_1}, \underbrace{b'_2, \dots, b'_2}_{Q_2} \dots \underbrace{b'_r \dots, b'_r}_{Q_r}),$$

and $a'_1 > \dots > a'_k$, $b'_1 > \dots > b'_r$. Moreover, $x' = \alpha x + \beta, y' = \alpha' y + \beta'$ for some $\alpha' \geq \alpha > 0, \beta' \geq \beta$.

Proof. From the definition of strategically equivalent interval games it follows that $\underline{w'} = \alpha\underline{w} + \bar{\beta}, \overline{w'} = \alpha'\overline{w} + \bar{\beta}'$, where $\bar{\beta}' \geq \bar{\beta}$, $\alpha' \geq \alpha > 0$, and $\alpha' > \alpha$ only if $w(S), w'(S) \geq 0$ for all $S \subset N$. Then formulas (8),(9) give the result. □

Monotonicity properties of the interval Dutta–Ray solution have been already discussed in the previous section. Now we are going to define and to show consistency properties of the interval Dutta–Ray solution.

5 Consistency of the Dutta-Ray solution on the class of convex interval games and its axiomatic characterization

Consistency properties of a solution connect the solution vectors of TU games with different sets of players. More exactly, a TU game solution σ is *consistent*, if, given a TU game $\langle N, v\rangle$ and a solution vector $x \in \sigma(N, v)$, for any coalition $S \subset N$ the vector $x_{N\setminus S}$ belongs to the solution $\sigma(N \setminus S, v^x)$ $(\sigma(N \setminus S, v^\sigma))$ of the *reduced game*, obtained from $\langle N, v\rangle$ after leaving the coalitions S. The characteristic function of the reduced game is defined in different ways depending on methods of aggregating the values $v(T \cup Q)$ for $T \subset N \setminus S$, $Q \subset S$ and x_S (or on the solution σ itself) into a unique characteristic function value $v^x_{N\setminus S}(T)$ $(v^\sigma_{N\setminus S})$ of the reduced game.

Thus, to consider consistency properties of a solution, we should put into consideration the classes of games with different sets of players. Let $\mathcal{N}$ be an arbitrary *universal* set of players, $G_N(IG_N)$ be an arbitrary class of TU (interval) games with the player set N. Denote by $G_{\mathcal{N}} = \bigcup_{N\subset\mathcal{N}} G_N,\ \ IG_{\mathcal{N}} = \bigcup_{N\subset\mathcal{N}} IG_N$ the classes of all TU games and interval games whose finite sets of players are contained in the universal set $\mathcal{N}$, and characteristic functions are defined by the classes $G_N, IG_N, N \subset \mathcal{N}$, respectively.

Dutta (1990) showed that the DR solution on the class of convex TU games $G^c_{\mathcal{N}}$ is consistent in the definition of Davis–Maschler (max consistency)(Davis and Maschler (1965)) and of Hart–Mas-Colell (self consistency) (Hart and Mas-Colell (1989)). We extend the definitions of consistency of TU game solutions to the generated by them interval solutions by demanding consistency of the corresponding TU game solutions for both border games. Since the Dutta–Ray solution is single-valued both for TU and interval convex games, we give the definitions of interval consistency in the definitions of Davis–Maschler and of Hart–Mas-Colell only for single-valued solutions.

A single-valued solution ϕ on a class $IG_{\mathcal{N}}$ of interval games generated by a TU game solution φ on a class $G^c_{\mathcal{N}}$ is *consistent* or satisfies the *reduced game property in the sense of Davis–Maschler* if for any game $\langle N, (\underline{w}, \overline{w})\rangle \in IG_{\mathcal{N}}$ and a coalition $S \subset N$

$$(\varphi(N, \underline{w}), \varphi(N, \overline{w}))_S = (\varphi(S, \underline{w}^x), \varphi(S, \overline{w}^y)), \tag{10}$$

where $x = \varphi(N, \underline{w})$, $y = \varphi(N, \overline{w})$, $\langle S, (\underline{w}^x, \overline{w}^y)\rangle \in IG_S$ and the characteristic functions of the upper and lower reduced games are defined as follows:

$$\underline{w}^x(T) = \begin{cases} \underline{w}(N) - x(N \setminus S), & \text{if } T = S, \\ \max\limits_{Q\subset N\setminus S}\Big(\underline{w}(T \cup Q) - x(Q)\Big) & \text{for other } T \subset S, \end{cases} \tag{11}$$

$$\overline{w}^y(T) = \begin{cases} \overline{w}(N) - y(N \setminus S), & \text{if } T = S, \\ \max\limits_{Q\subset N\setminus S}\Big(\overline{w}(T \cup Q) - y(Q)\Big) & \text{for other } T \subset S. \end{cases} \tag{12}$$

Moreover, the reduced interval games $\langle S, (\underline{w}^y, \overline{w}^x)\rangle$ should belong to the class $IG^c_{\mathcal{N}}$ for all $S \subset N$.

In definitions (11),(12) the characteristic functions of the reduced on S interval game depend on the solution payoffs $x_i, i \in N \setminus S$ of players leaving the game. Hart and Mas-Colell proposed another approach to the definition of reduced games where they depend on solutions of subgames of the initial game.

A solution ϕ on the class $IG^c_{\mathcal{N}}$ of interval games, generated by a TU game solution φ, is *consistent* or satisfies the *reduced game property in the sense of Hart–Mas-Colell* if for any game $\langle N, (\underline{w}, \overline{w})\rangle \in IG_{\mathcal{N}}$, a coalition $S \subset N$, it holds

$$(\varphi(N, \underline{w}), \varphi(N, \overline{w}))_S = (\varphi(S, \underline{w}^{\varphi}), \varphi(S, \overline{w}^{\varphi})), \tag{13}$$

where the reduced games $\langle S, \underline{w}^{\varphi}\rangle$, $\langle S, \overline{w}^{\varphi}\rangle \in IG^c_{\mathcal{N}}$ are defined as follows:

$$\begin{aligned} \underline{w}^{\varphi}(T) = &\ \ \underline{w}(T \cup (N \setminus S)) - \textstyle\sum_{j \in N \setminus S} \underline{\varphi}_j(T \cup (N \setminus S), \underline{w}), \\ \overline{w}^{y}(T) = &\ \ \overline{w}(T \cup (N \setminus S)) - \textstyle\sum_{j \in N \setminus S} \overline{\varphi}_j(T \cup (N \setminus S), \overline{w}), \end{aligned}$$

where $\langle T \cup (N \setminus S), \underline{w}\rangle, \langle T \cup (N \setminus S), \overline{w}\rangle$ are the subgames of the lower and upper games $\langle N, \underline{w}\rangle$, $\langle N, \overline{w}\rangle$, respectively.

An interval solution φ is *bilateral consistent* in the sense of Davis–Maschler (Hart–Mas-Colell) if equality (10) ((13)) holds only for two-person coalitions S, i.e. $|S| = 2$.

Since the given above definitions of consistency given above are applied separately to lower and upper games, it may seem that the results about consistency of TU games solutions can be directly extended to interval games. However, convex interval games demand convexity not only of lower and upper games, but also convexity of the length game. Just this property can be violated by the classical reduced games that does not permit to extend consistency of the DR solution to the interval setting.

Proposition 5 *The Dutta–Ray solution over the class* $IG^c_{\mathcal{N}}$ *with* $|\mathcal{N}| \geq 4$ *does not satisfy the Davis–Maschler consistency.*

Proof. We give an example of three-person convex interval game whose Davis–Maschler reduced interval games with respect to the DR solution do not belong to the class $IG^c_{\mathcal{N}}$.

Example 1 Let $N = \{1, 2, 3\}$. Consider the following interval game $\langle N, (\underline{w}, \overline{w})\rangle$:

$$\overline{w}(S) = \begin{cases} 3, & \text{if } S = \{1,2\} \\ 5, & \text{if } S = \{1,2,3\} \\ 0 & \text{for other } S; \end{cases},$$

$$\underline{w}(S) = \begin{cases} 3, & \text{if } S = \{1,2\}, \\ 4, & \text{if } S = \{1,2,3\} \\ 0 & \text{for other } S. \end{cases},$$

$$(\overline{w} - \underline{w})(S) = \begin{cases} 1, & \text{for } S = \{1,2,3\} \\ 0 & \text{for other } S. \end{cases},$$

Then $\overline{w}(S) \geq \underline{w}(S)$ for all $S \subset N$, and all games $\langle N, \underline{w}\rangle, \langle N, \overline{w}\rangle, \langle N, \overline{w} - \underline{w}\rangle$ are convex.

$$DR(N, \underline{w}) = (\frac{3}{2}, \frac{3}{2}, 1) = x, \;\; DR(N, \overline{w}) = (\frac{5}{3}, \frac{5}{3}, \frac{5}{3}) = y.$$

Consider the reduced games $\langle \{2,3\}, \overline{w}^y \rangle, \langle \{2,3\}, \underline{w}^x \rangle$ of the games $\langle N, \overline{w} \rangle, \langle N, \underline{w} \rangle$ on the player set $\{2,3\}$ and with respect to the payoff vectors y and x, respectively. Then

$$\begin{array}{rl} \overline{w}^y(2) = & \max\{0, 3 - \frac{5}{3}\} = \frac{4}{3}, \\ \underline{w}^x(2) = & \max\{0, 3 - \frac{3}{2}\} = \frac{3}{2}, \end{array}$$

and we obtain $\overline{w}^y(\{2\}) < \underline{w}^x(\{2\})$ that means the reduced interval game $\langle \{2,3\}, (\overline{w}^y, \underline{w}^x) \rangle \notin IG^c_{\mathcal{N}}$.

Let us consider the Hart–Mas-Colell consistency of the interval DR solution. To begin with we should return to the DR solution on the class of convex TU games $G^c_{\mathcal{N}}$. Dutta (1990) showed that the DR solution on the class of convex TU games $G^c_{\mathcal{N}}$ is consistent in the definition of Davis–Maschler and of Hart–Mas-Colell. However, he did not prove that the Hart–Mas-Colell reduced games of a convex TU game with respect to DR solution are convex. The following example shows that, in fact, they may be not convex:

Example 2 $N = \{1,2,3,4\}, \;\; v(\{i\}) = 0$ for all $i \in N$, $v(N) = 6 + 3\varepsilon$,
$v(\{1,2\}) = 4, v(\{1,3\}) = 1/2, v(\{i,j\}) = 1$ for other $(i,j) \neq (1,3)$,
$v(\{1,2,3\}) = 5 + 2\varepsilon, \;\; v(\{1,2,4\}) = 5 + \varepsilon, \; v(\{1,3,4\}) = v(\{2,3,4\}) = 2$.

Then for sufficiently small positive ε this game $\langle N, v \rangle$ is convex, and $DR(N, v) = (2, 2, 1 + 2\varepsilon, 1 + \varepsilon)$.

Consider the Hart–Mas-Colell reduced game $\langle N \setminus \{1\}, v^{DR} \rangle$ on the set $(2,3,4)$ with respect to the DR solution. Then
$v^{DR}(\{2\}) = 2, \; v^{DR}(\{3\}) = 1/4, \; v^{DR}(\{4\}) = 1/2$,
$v^{DR}(\{2,3\}) = 3 + 2\varepsilon, v^{DR}(\{2,4\}) = 3 + \varepsilon, \; v^{DR}(\{3,4\}) = 4/3, \; v^{DR}(\{2,3,4\}) = 4 + 3\varepsilon$,
and for $\varepsilon < 1/12$, it holds

$$v^{DR}(\{2,3\}) + v^{DR}(v\{3,4\}) = 4\frac{1}{3} + 2\varepsilon > 4\frac{1}{4} + 3\varepsilon = v^{DR}(\{2,3,4\}) + v^{DR}(\{3\}),$$

implying that the reduced game $\langle \{2,3,4\}, v^{DE} \rangle$ is not convex.

However, it is possible to establish the fact of bilateral Hart–Mas-Colell consistency of the DR solution:

Proposition 6 *The DR solution is bilateral consistent on the class $IG_{\mathcal{N}}$ for all $\mathcal{N}, |\mathcal{N}| \geq 3$.*

Proof. Let $\langle N, (\underline{w}, \overline{w}) \rangle \in IG^c_{\mathcal{N}}$ be an arbitrary game, $y = DR(N, \overline{w}), x = DR(N, \underline{w}), i, j \in N$. Consider the reduced game $\langle \{i,j\}, (\underline{w}^{DR}, \overline{w}^{DR}) \rangle$ on the set (i,j) with respect to the DR solution. Then by the definition of Hart–Mas-Colell consistency and the population monotonicity of the DR solution:

$$\begin{array}{rl} \overline{w}^{DR}(\{i\}) & = DR_i(N \setminus \{j\}, \overline{w}) \leq y_i, \\ \overline{w}^{DR}(\{j\}) & = DR_j(N \setminus \{i\}, \overline{w}) \leq y_j, \\ \overline{w}^{DR}(\{i,j\}) & = y_i + y_j. \end{array} \qquad (14)$$

From (14) it follows that the reduced game is superadditive and, hence, convex. Similarly it is proved that the reduced game $\langle\{i,j\},\underline{w}^{DR}\rangle$ and the length game $\langle\{i,j\},(\overline{w}^{DR}-\underline{w}^{DR})\rangle$ are both superadditive.

Let us show that $\underline{w}^{DR}\leq\overline{w}^{DR}$. By Proposition 1 providing the existence of the interval DR solution, we have

$$\underline{w}^{DR}(\{i\})=DR_i(N\setminus\{j\},\underline{w})\leq DR_i(N\setminus\{j\},\overline{w})=\overline{w}^{DR}(\{i\}).$$

The same equalities and inequality hold when we interchange i with j. At last,

$$\underline{w}^{DR}(\{i,j\})=x_i+x_j\leq y_i+y_j=\overline{w}^{DR}(\{i,j\}).$$

Thus, the reduced game on the two-player set $\{i,j\}$ belongs to the class $IG^c_{\mathcal{N}}$. The bilateral Hart–Mas-Colell consistency of the DR solution on the class of convex TU games implies the equalities $(x_i,x_j)=DR(\{i,j\},\underline{w}^{DR})$, $(y_i,y_j)=DR(\{i,j\},\overline{w}^{DR})$ proving the proposition. □

It turns out that bilateral consistency à la Hart–Mas-Colell of the interval DR solution together with the CE solution on two-person convex interval games are sufficient for the characterization of the interval DR solution on the class IG^c_N. To establish this result, first, let us prove an auxiliary one.

Lemma 1 *If a single-valued solution φ on the class G^c of convex TU games is bilateral consistent á la Hart–Mas-Colell and coincides with the solution of constrained egalitarianism on the class of two-person superadditive games, then it is efficient and belongs to the core.*

Proof. First, let us show efficiency of φ. Let $\langle N,v\rangle\in\mathcal{G}^c$ be an arbitrary game and let $y=\varphi(N,v)$. By efficiency of the solution of constrained egalitarianism, bilateral consistency of φ, and the definition of the Hart–Mas-Colell reduced games for any $i,j\in N$, we have

$$y_i+y_j=v^{\varphi}(\{i,j\})=v(N)-\sum_{k\in N\setminus\{i,j\}}\varphi_k(N,v)=v(N)-\sum_{k\neq i,j}y_k, \tag{15}$$

where $\langle\{i,j\},v^{\varphi}\rangle$ is the Hart–Mas-Colell reduced game on the player set $\{i,j\}$ with respect to the solution φ. From (15) it follows $\sum_{i\in N}y_i=v(N)$.

The next claim is to prove that $y\in C(N,\overline{w})$. We will prove the claim by induction in the number of players.

For two-person games we have $CE=\varphi$ and, hence, $\varphi(\{i,j\},v)\in C(\{i,j\},v)$. Assume that the claim is valid for all convex TU games whose number of players is less than $|N|$.

By bilateral consistency of φ, for every $i,j\in N$,

$$y_i\geq v^{\varphi}_{\{i,j\}}(\{i\})=\varphi_i(N\setminus\{j\},v). \tag{16}$$

By the inductive hypothesis equality (16) implies $y(S)\geq v(S)$ for all S, $|S|\leq n-1$. For $S=N$ efficiency of φ gives $y(N)=v(N)$ and we obtain $y\in C(N,v)$. □

Now we are ready to obtain an axiomatic characterization of the interval DR solution on the class of convex interval games.

Theorem 1 *For arbitrary universal set $\mathcal{N}$ the Dutta–Ray solution is the unique solution on the class $IG^c_{\mathcal{N}}$ satisfying constrained egalitarianism for two-person games and bilateral consistency á la Hart–Mas-Colell.*

Proof. In view of Proposition 6 only the uniqueness should be proved. Let φ be an arbitrary solution on the class $IG^c_{\mathcal{N}}$ satisfying the properties given in the Theorem, and for an arbitrary interval game $\langle N, (\underline{w}, \overline{w})\rangle \in IG^c_{\mathcal{N}}$ $y = \varphi(N, \overline{w})$, $x = \varphi(N, \underline{w})$.

Let us prove the equalities, $y = DR(N, \overline{w})$, $x = DR(N, \underline{w})$. It suffices to prove only one equality, the second one is proved analogously. Note that by Lemma 1 $y \in C(N, \overline{w})$.

Consider the following cases:

1^0. $y_i = y_j = \frac{\overline{w}(N)}{|N|}$ for all $i, j \in N$. Since $y \in C(N, \overline{w})$, this vector Lorenz dominates all other vectors from the core, that yields $y = DR(N, \overline{w})$.

2^0. There are $i, j \in N$ such that $y_i > y_j$. Represent y in the form

$$y = (\underbrace{y_1, \ldots, y_1}_{Q_1}, \underbrace{y_2, \ldots, y_2}_{Q_2}, \ldots \underbrace{y_l \ldots, y_l}_{Q_l}), \quad \text{where } y_1 > y_2 > \ldots > y_l,$$

and

$$DR(N, \overline{w}) = z = (\underbrace{z_1, \ldots, z_1}_{T_1}, \underbrace{z_2, \ldots, z_2}_{T_2}, \ldots \underbrace{z_m \ldots, z_m}_{T_m}), \quad \text{where } z_1 > z_2 \ldots > z_m.$$

Then by bilateral consistency of $\overline{\varphi}$ and the definition of constrained egalitaranism, for each $i \in N \setminus Q_l$ and $j \in Q_l$

$$y_i = \overline{w}^{\varphi}_{\{i,j\}}(\{i\}) = \overline{\varphi}_i(N \setminus \{j\}, \overline{w}), \tag{17}$$

where $\langle \{i, j\}, \overline{w}^{\varphi}_{\{i,j\}}\rangle$ is the Hart–Mas-Colell reduced game on the player set $\{i, j\}$.

By the inductive hypothesis equality (17) implies

$$y_i = DR_i(N \setminus \{j\}, \overline{w}) \text{ for each } i \in N \setminus Q_l, j \in Q_l. \tag{18}$$

Let us show that $T_1 \cap Q_l = \emptyset$. In fact, equality (18) and population monotonicity of the DR solution imply $y_i \le z_1$ for all $N \setminus Q_1$, and $y_j < y_i$ for such i and $j \in Q_l$. Therefore, if $T_1 \cap Q_l \neq \emptyset$, then $y(T_1) = \sum_{i \in T_1} y_i < z_1|T_1| = \overline{w}(T_1)$, that would contradict the membership of $y = \varphi(N, \overline{w})$ to the core.

Thus, we have obtained the equalities

$$y_i = \overline{\varphi}_i(N \setminus \{j\}, \overline{w}) = DR_i(N \setminus \{j\}, \overline{w}) = DR_i(N, \overline{w}) = z_1 \text{ for all } i \in T_1, j \in Q_l.$$

Consider the following possibilities:

2^0a. $T_1 \cup Q_l = N$. If $m = 2$, then $y = z$, and the proof is complete.

If $m > 2$, then $Q_l = T_2, \cup ... \cup T_m$, and for $i \in T_k, j \in T_l, k < l,\ k, l = 2, ..., m$, we have $DR_i(N, \overline{w}) = z_k > z_l = DR_j(N, \overline{w})$. Let $k \in \{2, \ldots, m\}$ be a number such that

$$\begin{aligned} z_r >\ & y_l \text{ for } r < k \\ z_r \le\ & y_l \text{ for } r \ge k. \end{aligned}$$

Such a k does exist because $y(Q_l) = z(Q_l)$ and $y_j = y_l$ for all $j \in Q_l$. Denote $Z_k = \bigcup_{t=1}^k T_t$. Then by the definition of the DR solution and by the equalities $y_j = z_j$ for $j \in T_1$,

$$\overline{w}(R_k) = z(R_k) = z_1|T_1| + \sum_{t=2}^{k} z_t|T_t| > y(T_1) + y_l \sum_{t=2}^{k} |T_j|.$$

that again would contradict the membership of y to the core $C(N, \overline{w})$, Thus, the case $T_1 \cap Q_l = \emptyset, m > 2$ is impossible and we return to the case $m = 2$.

2^0b. $T_1 \cup Q_l \subsetneqq N$. Repeat the procedure for the set T_2. First, let us show that $T_2 \cap Q_l = \emptyset$. As in the proof of the previous case, equality (18) and population monotonicity of the DR solution imply $y_i \leq z_1$ for all $N \backslash Q_1$, and $y_j < y_i$ for such i and $j \in Q_l$. Therefore, if $T_2 \cap Q_l \neq \emptyset$, then $y(T_2) = \sum_{i \in T_2} y_i < z_2 |T_2|$, and this inequality together with the proven equality $y(T_1) = z(T_1)$ yield

$$y(T_1 \cup T_2) < z(T_1 \cup T_2) = \sum_{i \in T_1 \cup T_2} DR_i(N, \overline{w}) = \overline{w}(T_1 \cup T_2),$$

that would contradict the membership of y to the core $C(N, \overline{w})$.

Hence, $T_2 \cap Q_l = \emptyset$, implying that for any $i \in T_2, j \in Q_l$

$$y_i = \overline{\varphi}(N \setminus \{j\}, \overline{w}) = DR_i(N \setminus \{j\}, \overline{w}) = DR_i(N, \overline{w}) = z_2,$$

and we obtain the equality $z_{T_2} = y_{T_2}$. If $m = 3$, then the process finished and $z = y$. If $m > 3$, then we again repeat the procedure, and in the $(m-1)$-th step we obtain $y = z$, that completes the proof. $\square$

References

[1] Alparslan Gök S.Z., Branzei R. and Tijs S., "Cores and stable sets for interval-valued games", *preprint no.* ***90****, Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 17* (2008a).

[2] Alparslan Gök S.Z., Branzei R. and Tijs S., "Convex interval games", *preprint no.* ***100****, Institute of Applied Mathematics, METU* and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 37* (2008b).

[3] Alparslan Gök S.Z., Miquel S. and Tijs S., "Cooperation under interval uncertainty", *preprint no.* ***73****, Institute of Applied Mathematics, METU* (2007) and *Tilburg University, Center for Economic Research, The Netherlands, CentER DP 09* (2008) (to appear in Mathematical Methods of Operations Research,
DOI number: `http://dx.doi.org/10.1007/s00186-008-0211-3`).

[4] Branzei R., Tijs S. and Alparslan Gök S.Z., "How to handle interval solutions for cooperative interval games", *preprint no.* ***110****, Institute of Applied Mathematics, METU* (2008).

[5] Charnes A. and Granot D., "Prior solutions: extensions of convex nucleolus solutions to chance-cnstrained games", *Proceedings of the Computer Science and Statistics Seventh Symposium at Iowa State University* (1973) 323-332.

[6] Davis M. and Maschler M., "The kernel of a cooperative game", *Naval Research Logistic Quarterly* **12** (1965).

[7] Dutta B., "The egalitarian solution and the reduced game properties in convex games", *International Journal of Game Theory* **19** (1990) 153-159.

[8] Dutta B. and Ray D., "A concept of egalitarianism under participation constraints", *Econometrica* **57** (1989) 615-630.

[9] Hart S. and Mas-Colell A., "Potential, value, and, consistency", *Econometrica* **57** (1989) 589-614.

[10] Hokari T., "The nucleolus is not aggregate monotonic on the domain of convex games", *International Journal of Game Theory* **29** (2000) 133-137.

[11] Hokari T. and van Gellekom A., "Population monotonicity and consistency in convex games: Some logical relations", *International Journal of Game Theory* **31** (2002) 593-607.

[12] Suijs J., Borm P., De Waegenaere A. and Tijs S., "Cooperative games with stochastic payoffs", *European Journal of Operational Research* **113** (1999) 193-205.

[13] Sprumont Y., "Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility", *Games and Economic Behavior* **2** (1990) 378-394.

[14] Timmer J., Borm P. and Tijs S., "Convexity in stochastic cooperative situations", *International Game Theory Review* **7** (2005) 25-42.